

# About some mappings in $\lambda(r)$ -regular metric spaces

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## Abstract

It is formulated conditions on functions  $Q(x)$  and boundaries of domains under which every  $Q$ -homeomorphism admits continuous or homeomorphic extension to the boundary in metric spaces with measures.

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## 1 Introduction

Mapping theory started in the 18th century. Beltrami, Caratheodory, Christoffel, Gauss, Hilbert, Liouville, Poincaré, Riemann, Schwarz, and so on all left their marks in this theory. Conformal mappings and their applications to potential theory, mathematical physics, Riemann surfaces, and technology played a key role in this development.

During the late 1920s and early 1930s, Grötzsch, Lavrentiev, and Morrey introduced a more general and less rigid class of mappings that were later named quasiconformal. The concept of  $Q$ -homeomorphism is a natural extension of the geometric definition of quasiconformality; see, e.g., [1]. The subject of  $Q$ -homeomorphisms is interesting on its own right and has applications to many classes of mappings. In particular, the theory of  $Q$ -homeomorphisms can be applied to mappings in local Sobolev classes (see, e.g., Sections 6.3 and 6.10 in [1]) to the mappings with finite length distortion (see Sections 8.6 and 8.7 in [1]) and to the finitely bi-Lipschitz mappings; see Section 10.6 in [1]. The main goal of the theory of  $Q$ -homeomorphisms is to clear up various interconnections between properties of the majorant  $Q(x)$  and the corresponding properties of the mappings themselves.

## 2 Preliminaries

Given a set  $S$  in  $(X, d)$  and  $\alpha \in [0, \infty)$ ,  $H^\alpha$  denotes the  **$\alpha$ -dimensional Hausdorff measure** of  $S$  in  $(X, d)$ , i.e.,

$$H^\alpha(S) = \sup_{\varepsilon > 0} H_\varepsilon^\alpha(S), \quad (2.1)$$

$$H_\varepsilon^\alpha(S) = \inf \sum_{i=1}^{\infty} \delta_i^\alpha, \quad (2.2)$$

where the infimum is taken over all countable collections of numbers  $\delta_i \in (0, \varepsilon)$  such that some sets  $S_i$  in  $(X, d)$  with diameters  $\delta_i$  cover  $S$ . Note that  $H^\alpha$  is nonincreasing in the parameter  $\alpha$ . The **Hausdorff dimension** of  $S$  is the only number  $\alpha \in [0, \infty]$  such that  $H^{\alpha'}(S) = 0$  for all  $\alpha' > \alpha$  and  $H^{\alpha''}(S) = \infty$  for all  $\alpha'' < \alpha$ .

Recall, for a given continuous path  $\gamma : [a, b] \rightarrow X$  in a metric space  $(X, d)$ , that its **length** is the supremum of the sums

$$\sum_{i=1}^k d(\gamma(t_i), \gamma(t_{i-1}))$$

over all partitions  $a = t_0 \leq t_1 \leq \dots \leq t_k = b$  of the interval  $[a, b]$ . The path  $\gamma$  is called **rectifiable** if its length is finite.

In what follows,  $(X, d, \mu)$  denotes a space  $X$  with a metric  $d$  and a locally finite Borel measure  $\mu$ . Given a family of paths  $\Gamma$  in  $X$ , a Borel function  $\varrho : X \rightarrow [0, \infty]$  is called **admissible** for  $\Gamma$ , abbr.  $\varrho \in \text{adm } \Gamma$ , if

$$\int_{\gamma} \varrho ds \geq 1 \quad (2.3)$$

for all  $\gamma \in \Gamma$ .

An open set in  $X$  whose points can all be connected pairwise by continuous paths is called a **domain** in  $X$ . Let  $D$  and  $D'$  be domains with finite Hausdorff dimensions  $\alpha$  and  $\alpha' \geq 1$  in spaces  $(X, d, \mu)$  and  $(X', d', \mu')$ , and let  $Q : D \rightarrow [0, \infty]$  be a measurable function. We say that a homeomorphism  $f : D \rightarrow D'$  is a  **$Q$ -homeomorphism** if

$$M(f\Gamma) \leq \int_D Q(x) \cdot \varrho^\alpha(x) d\mu(x) \quad (2.4)$$

for every family  $\Gamma$  of paths in  $D$  and every admissible function  $\varrho$  for  $\Gamma$ .

The **modulus** of the path family  $\Gamma$  in  $D$  is given by the equality

$$M(\Gamma) = \inf_{\varrho \in \text{adm } \Gamma} \int_D \varrho^\alpha(x) d\mu(x). \quad (2.5)$$

In the case of the path family  $\Gamma' = f\Gamma$ , we take the Hausdorff dimension  $\alpha'$  of the domain  $D'$ .

A space  $(X, d, \mu)$  is called **(Ahlfors)  $\alpha$ -regular** if there is a constant  $C \geq 1$  such that

$$C^{-1}r^\alpha \leq \mu(B_r) \leq Cr^\alpha \quad (2.6)$$

for all balls  $B_r$  in  $X$  with the radius  $r < \text{diam } X$ . A space  $(X, d, \mu)$  is **(Ahlfors) regular** if it is (Ahlfors)  $\alpha$ -regular for some  $\alpha \in (1, \infty)$ .

A space  $(X, d, \mu)$  is **upper  $\alpha$ -regular at a point**  $x_0 \in X$  if there is a constant  $C > 0$  such that

$$\mu(B(x_0, r)) \leq Cr^\alpha \quad (2.7)$$

for the balls  $B(x_0, r)$  centered at  $x_0 \in X$  with all radii  $r < r_0$  for some  $r_0 > 0$ . A space  $(X, d, \mu)$  is **upper  $\alpha$ -regular** if condition (2.7) holds at every point  $x_0 \in X$ , see [1], p. 258.

Recall that a topological space is **connected space** if it is impossible to split it into two non-empty open sets. Compact connected spaces are called **continua**. A topological space  $T$  is said to be **path connected** if any two points  $x_1$  and  $x_2$  in  $T$  can be joined by a path  $\gamma : [0, 1] \rightarrow T$ ,  $\gamma(0) = x_1$  and  $\gamma(1) = x_2$ . A **domain** in  $T$  is an open path connected set in  $T$ . A domain  $D$  in a topological space  $T$  is called **locally connected at a point**  $x_0 \in \partial D$  if, for every neighborhood  $U$  of the point  $x_0$ , there is its neighborhood  $V \subseteq U$  such that  $V \cap D$  is connected, [2], c. 232. Similarly, we say that a domain  $D$  is **locally path connected at a point**  $x_0 \in \partial D$  if, for every neighborhood  $U$  of the point  $x_0$ , there is its neighborhood  $V \subseteq U$  such that  $V \cap D$  is path connected.

The boundary of  $D$  is **weakly flat at a point**  $x_0 \in \partial D$  if, for every number  $P > 0$  and every neighborhood  $U$  of the point  $x_0$ , there is its neighborhood  $V \subset U$  such that

$$M(\Delta(E, F; D)) \geq P \quad (2.8)$$

for all continua  $E$  and  $F$  in  $D$  intersecting  $\partial U$  and  $\partial V$ .

The boundary of the domain  $D$  is **strongly accessible at a point**  $x_0 \in \partial D$ , if, for every neighborhood  $U$  of the point  $x_0$ , there is a compact set  $E \subset D$ , a neighborhood  $V \subset U$  of the point  $x_0$  and a number  $\delta > 0$  such that

$$M(\Delta(E, F; D)) \geq \delta \quad (2.9)$$

for every continuum  $F$  in  $D$  intersecting  $\partial U$  and  $\partial V$ .

Finally, we say that the boundary  $\partial D$  is **weakly flat** and **strongly accessible** if the corresponding properties hold at every point of the boundary, see [1].

We start first from the following general statement, see Lemma 13.3 and Theorem 13.3 in [1].

**Lemma 2.1.** *Let a space  $X$  be path connected at a point  $x_0 \in D$  that has a compact neighborhood, let  $X'$  be a compact weakly flat space, and let  $f : D \rightarrow D'$*

be a  $Q$ -homeomorphism, where  $Q : D \rightarrow [0, \infty]$  is a measurable function satisfying the condition

$$\int_{D \cap A(x_0, \varepsilon, \varepsilon_0)} Q(x) \cdot \psi_{x_0, \varepsilon}^\alpha(d(x, x_0)) d\mu(x) = o(I_{x_0}^\alpha(\varepsilon)) \quad (2.10)$$

as  $\varepsilon \rightarrow 0$ ,  $A(x_0, \varepsilon, \varepsilon_0) = \{x \in X : \varepsilon < d(x, x_0) < \varepsilon_0\}$ , where  $\varepsilon_0 < \text{dist}(x_0, \partial D)$  and  $\psi_{x_0, \varepsilon}(t)$  is a family of nonnegative (Lebesgue) measurable functions on  $(0, \infty)$  such that

$$0 < I_{x_0}(\varepsilon) = \int_{\varepsilon}^{\varepsilon_0} \psi_{x_0, \varepsilon}(t) dt < \infty, \quad \varepsilon \in (0, \varepsilon_0). \quad (2.11)$$

Then  $f$  can be extended to the point  $x_0$  by continuity in  $X'$ .

**Theorem 2.1.** Let  $D$  be locally path connected at all its boundary points and  $\overline{D}$  compact,  $D'$  with a weakly flat boundary, and let  $f : D \rightarrow D'$  be a  $Q$ -homeomorphism with  $Q \in L_\mu^1(D)$ . Then the inverse homeomorphism  $g = f^{-1} : D' \rightarrow D$  admits a continuous extension  $\overline{g} : \overline{D'} \rightarrow \overline{D}$ .

### 3 Basic results

We will say that a space  $(X, d, \mu)$  is **upper  $\lambda(r)$ -regular at a point**  $x_0 \in X$  if there is a constant  $C > 0$  such that

$$\mu(B(x_0, r)) \leq C \lambda_{x_0}(r) \quad (3.1)$$

for the balls  $B(x_0, r)$  centered at  $x_0 \in X$  with all radii  $r < r_0$  for some  $r_0 > 0$ , and  $\lambda(r)$  is increasing function. We will also say that a space  $(X, d, \mu)$  is **upper  $\lambda(r)$ -regular** if condition (3.1) holds at every point  $x_0 \in X$ .

**Lemma 3.1.** Let  $D$  be a domain in a space  $(X, d, \mu)$  that is upper  $\lambda(r)$ -regular at the point  $x_0 \in \overline{D}$  and  $\lambda$  is increasing function. If for every nonnegative function  $\varphi : D \rightarrow \mathbb{R}$  condition

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{\mu(B(x_0, \varepsilon) \cap D)} \int_{B(x_0, \varepsilon) \cap D} |\varphi(x)| d\mu(x) < \infty \quad (3.2)$$

holds then

$$\int_{D \cap A(x_0, \varepsilon, \varepsilon_0)} \frac{|\varphi(x)| d\mu(x)}{\lambda(d(x, x_0))} = O\left(\log \frac{1}{\varepsilon}\right) \quad (3.3)$$

as  $\varepsilon \rightarrow 0$  and some  $\varepsilon_0 \in (0, \delta_0)$ , where  $\delta_0 = \min(e^{-1}, d_0)$ ,  $d_0 = \sup_{x \in D} d(x, x_0)$ ,  $A(x_0, \varepsilon, \varepsilon_0) = \{x \in X : \varepsilon < d(x, x_0) < \varepsilon_0\}$ ,  $B_r = B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ .

*Proof.* Choose  $\varepsilon_0 \in (0, \delta_0)$  such that the function  $\varphi$  is integrable in  $D$  with respect to the measure  $\mu$ , where

$$\delta_0 = \sup_{r \in (0, \varepsilon_0)} \frac{1}{\mu(D_r)} \int_{D_r} |\varphi(x)| d\mu(x) < \infty,$$

$D_r = D \cap B_r$ . Further, let  $\varepsilon < 2^{-1}\varepsilon_0$ ,  $\varepsilon_k = 2^{-k}\varepsilon_0$ ,  $A_k = \{x \in X : \varepsilon_{k+1} \leq d(x, x_0) < \varepsilon_k\}$ ,  $B_k = B(\varepsilon_k)$ . Choose a natural number  $N$  such that  $\varepsilon \in [\varepsilon_{N+1}, \varepsilon_N)$ . Then  $D \cap A(x_0, \varepsilon, \varepsilon_0) \subset \Delta(\varepsilon) := \bigcup_{k=0}^N \Delta_k$ , where  $\Delta_k = D \cap A_k$  and

$$\begin{aligned} \eta(\varepsilon) &= \int_{\Delta(\varepsilon)} \frac{|\varphi(x)| d\mu(x)}{\lambda_{x_0}(d(x, x_0))} \leq \sum_{k=1}^N \int_{\Delta_k} \frac{|\varphi(x)| d\mu(x)}{\lambda_{x_0}(d(x, x_0))} \leq \\ &\leq \sum_{k=1}^N \frac{1}{\lambda(\varepsilon_{k+1})} \int_{B(\varepsilon_k) \cap D} |\varphi(x)| d\mu(x) \leq \delta_0 \cdot \sum_{k=1}^N \frac{\mu(B_k \cap D)}{\lambda_{x_0}(\varepsilon_{k+1})}. \end{aligned}$$

In pursuance of upper  $\lambda(r)$ -regularity we obtain that  $\mu(B_k) \leq C \cdot \lambda_{x_0}(\varepsilon_k)$  and

$$\eta(\varepsilon) \leq C \cdot \delta_0 \cdot \sum_{k=1}^N \frac{\lambda_{x_0}(\varepsilon_k)}{\lambda_{x_0}(\varepsilon_{k+1})} \leq C \cdot \delta_0 \cdot N.$$

Since  $N < \log_2 \frac{1}{\varepsilon} = \frac{\log \frac{1}{\varepsilon}}{\log 2}$ , see [1] p.266, then

$$\int_{D \cap A(x_0, \varepsilon, \varepsilon_0)} \frac{|\varphi(x)| d\mu(x)}{\lambda_{x_0}(d(x, x_0))} \leq \frac{C \cdot \delta_0}{\log 2} \log \frac{1}{\varepsilon}.$$

So, we complete the proof.

As before, here  $(X, d, \mu)$  and  $(X', d', \mu')$  are spaces with metrics  $d$  and  $d'$  and locally finite Borel measures  $\mu$  and  $\mu'$ , and  $D$  and  $D'$  are domains in  $X$  and  $X'$  with finite Hausdorff dimensions  $\alpha$  and  $\alpha' > 1$ , respectively.

**Theorem 3.1.** *Let  $X$  be upper  $\lambda(r)$ -regular at a point  $x_0 \in \partial D$  where  $D$  is locally path connected,  $\overline{D'}$  be compact and  $\partial D'$  strongly accessible with*

$$\lambda(\varepsilon) = o\left(\varepsilon^\alpha \log^{\alpha-1} \frac{1}{\varepsilon}\right) \quad (3.4)$$

as  $\varepsilon \rightarrow 0$ . If

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{\mu(B(x_0, \varepsilon) \cap D)} \int_{B(x_0, \varepsilon) \cap D} Q(x) d\mu(x) < \infty, \quad (3.5)$$

then any  $Q$ -homeomorphism  $f : D \rightarrow D'$  can be extended to the point  $x_0$  by continuity in  $(X', d')$ .

*Proof.* By elementary argument, we see that condition (3.4) implies that

$$\int_0^{\varepsilon_0} \frac{dt}{\lambda^{\frac{1}{\alpha}}(t)} = \infty. \quad (3.6)$$

Choosing in Lemma 2.1  $\psi(t) = \frac{1}{\lambda^{1/\alpha}(t)}$  and combining conclusion Lemma 3.1, we show, by L'Hospital rule, that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{D \cap A(x_0, \varepsilon, \varepsilon_0)} \frac{Q(x) d\mu(x)}{\lambda(d(x, x_0))} / \left( \int_{\varepsilon}^{\varepsilon_0} \frac{dt}{\lambda^{1/\alpha}(t)} \right)^{\alpha} &\leq \lim_{\varepsilon \rightarrow 0} c \log \frac{1}{\varepsilon} / \left( \int_{\varepsilon}^{\varepsilon_0} \frac{dt}{\lambda^{1/\alpha}(t)} \right)^{\alpha} = \\ &= \gamma \lim_{\varepsilon \rightarrow 0} \frac{\lambda(\varepsilon)}{\varepsilon^{\alpha}} \log^{1-\alpha} \frac{1}{\varepsilon}, \end{aligned}$$

where  $\gamma = \left(\frac{c}{\alpha}\right)^{\alpha}$ . According to (3.4)

$$\lim_{\varepsilon \rightarrow 0} \int_{D \cap A(x_0, \varepsilon, \varepsilon_0)} \frac{Q(x) d\mu(x)}{\lambda(d(x, x_0))} / \left( \int_{\varepsilon}^{\varepsilon_0} \frac{dt}{\lambda^{1/\alpha}(t)} \right)^{\alpha} = 0.$$

Since conditions of Lemma 2.1 are hold then exist extending to  $x_0$  by continuity.

Combining the theorems (2.1) and (3.1), we obtain the following theorem.

**Theorem 3.2.** *Let  $X$  be upper  $\lambda(r)$ -regular at a point  $x_0 \in \partial D$  where  $D$  and  $D'$  have weakly flat boundaries, let  $\overline{D}$  and  $\overline{D}'$  be compact, and let  $Q : D \rightarrow [0, \infty]$  be a function of the class  $L_{\mu}^1(D)$  with condition (3.5). If*

$$\lambda(\varepsilon) = o \left( \varepsilon^{\alpha} \log^{\alpha-1} \frac{1}{\varepsilon} \right) \quad (3.7)$$

*as  $\varepsilon \rightarrow 0$  then any  $Q$ -homeomorphism  $f : D \rightarrow D'$  is extended to a homeomorphism  $\bar{f} : \overline{D} \rightarrow \overline{D}'$ .*

## References

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